

Magnetic monopoles and millicharged particles

Rhys Davies June 1, 2010

There is an old argument, due to Dirac, that the existence of magnetic monopoles implies electric charge quantisation. It may not be immediately clear how to reconcile this with the occurrence of ‘millicharged particles’ when one allows for kinetic mixing between the photon and a hidden $U(1)$ gauge boson. These notes are intended to clarify the issue.

1 Magnetic monopoles and charge quantisation

The question of whether or not electric charge is quantised or not is really a question about whether the gauge group is $U(1)$ or its non-compact version, isomorphic to the real line $\mathbb{R}$. To see this, consider a gauge transformation acting on two charged fields:

$$(\psi_1, \psi_2) \rightarrow (e^{i\alpha q_1} \psi_1, e^{i\alpha q_2} \psi_2) \quad (1)$$

We now ask whether the gauge parameter α is a periodic variable. It is easy to see that this is the case if and only if $q_1/q_2 \in \mathbb{Q}$ for each pair of charged fields¹. We can always then rescale α to have period 2π , in which case all charges q_i are integral i.e. charge is quantised.

Suppose that the gauge group is truly $U(1)$, and consider the existence of magnetic monopoles. Take some fixed two-sphere S^2 , and cover it by two patches $\mathcal{U}_N, \mathcal{U}_S$ which are the ‘northern’ and ‘southern’ hemispheres, each including the equator². If the coupling constant is e , then the corresponding gauge potentials A_N, A_S must be related by a gauge transformation on the equator:

$$A_N = A_S + \frac{1}{e} f^{-1} df \quad (2)$$

for some function $f : S^1 \rightarrow U(1)$. Such functions fall into homotopy classes, and the conclusions depend only on the homotopy class, so we will take f to be given by

$$f(\phi) = e^{in\phi}, \quad n \in \mathbb{Z} \quad (3)$$

¹Assuming that there are a finite number of such fields.

²Technically we should take $\mathcal{U}_N$ and $\mathcal{U}_S$ to overlap on some open neighbourhood of the equator, but such pedantry would only obfuscate the argument given here.

where ϕ is the azimuthal angle. Now we can measure how much magnetic charge is enclosed by our S^2 by integrating the magnetic flux:

$$\begin{aligned} g &= \int_{S^2} F = \int_{\mathcal{U}_N} dA_N + \int_{\mathcal{U}_S} dA_S \\ &= \int_{S^1} (A_N - A_S) = \frac{2\pi n}{e} \end{aligned} \quad (4)$$

We therefore obtain Dirac's quantisation condition $eg/2\pi \in \mathbb{Z}$. It's also clear that monopoles can exist if and only if charge is quantised, since the fundamental group of $\mathbb{R}$ is trivial, so we cannot obtain a non-zero magnetic charge.

1.1 Multiple gauge groups

Now suppose that the gauge group of the theory is $U(1) \times U(1)$, and denote by $\tilde{B}$ the gauge field of the second, 'hidden', factor, with corresponding field strength $\tilde{G}$. We will also add a tilde to the fields we had, anticipating a rescaling later. A monopole can now be charged with respect to each of the $U(1)$ factors:

$$\tilde{g} = \int_{S^2} \tilde{F} = \frac{2\pi n}{\tilde{e}} \quad , \quad \tilde{g}_h = \int_{S^2} \tilde{G} = \frac{2\pi m}{e_h} \quad (5)$$

The fun starts when we consider the Lagrangian of the theory, including a possible kinetic mixing term

$$\mathcal{L} = -\frac{1}{4}(\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} + \tilde{G}^{\mu\nu}\tilde{G}_{\mu\nu} + 2\chi\tilde{F}^{\mu\nu}\tilde{G}_{\mu\nu}) + \tilde{e}\tilde{A}_\mu J_A^\mu + e_h\tilde{B}_\mu J_B^\mu \quad (6)$$

where J_A and J_B are the currents corresponding to the two $U(1)$ symmetries. The physics is easiest to interpret with canonical kinetic terms, so to this end let $B = \tilde{B} + \chi\tilde{A}$ and $A = \sqrt{1 - \chi^2}\tilde{A}$ to obtain

$$\mathcal{L} = -\frac{1}{4}(F^{\mu\nu}F_{\mu\nu} + G^{\mu\nu}G_{\mu\nu}) + \frac{\tilde{e}}{\sqrt{1 - \chi^2}}A_\mu J_A^\mu + e_h(B_\mu - \frac{\chi}{\sqrt{1 - \chi^2}}A_\mu)J_B^\mu \quad (7)$$

We see that the only effect of the mixing on the 'visible' sector is to rescale the coupling constant to $e = \tilde{e}/\sqrt{1 - \chi^2}$, but particles in the hidden sector now interact with the visible photon, and typically have non-integral (indeed, irrational) charges. Experimentally, these charges (and therefore χ) must be tiny, otherwise such particles would have been observed.

We can now ask what the magnetic charges are with respect to the ‘physical’ gauge fields, with canonical kinetic terms. Plugging the field redefinitions into (5), we get

$$\begin{aligned} g &= \int_{S^2} F = \frac{2\pi n \sqrt{1-\chi^2}}{\tilde{e}} = \frac{2\pi n}{e} \\ g_h &= \int_{S^2} G = \frac{2\pi m}{e_h} + \chi \frac{2\pi n}{\tilde{e}} = 2\pi \left(\frac{m}{e_h} + \frac{\chi}{\sqrt{1-\chi^2}} \frac{n}{e} \right) \end{aligned} \tag{8}$$

There are two things to see here. One is that Dirac’s condition still holds for the charges of the ‘visible sector’ particles and monopoles. The other is that the magnetic monopoles of the visible sector now have a small magnetic charge under the hidden gauge group.

It is now clear that if the photon mixes with a hidden $U(1)$ gauge boson, there is no contradiction between the existence of magnetic monopoles and millicharged particles. The point is that the millicharged particles are really charged with respect to the linear combination $B - (\chi/\sqrt{1-\chi^2})A$. The corresponding magnetic charge is, from (8)

$$g' = \int_{S^2} \left(G - \frac{\chi}{\sqrt{1-\chi^2}} F \right) = \frac{2\pi m}{e_h}$$

So, as expected, there is no contradiction with Dirac’s condition.